ELASTIC AND PLASTIC BEHAVIOR OF CYLINDRICAL SHELLS

UNDER DYNAMIC LOADS BASED ON ENERGY CRITERIA

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ERRATA AND ADDENDUM

"Elastic and Plastic Behavior of Cylindrical Shells Under Dynamic Loads Based on Energy Criteria", J.E.Greenspon (J G Engineering Research Associates) Contract No. DA 36-034-ORD-3081RD, Tech. Rep. No. 3, Feb. 1963.

p. 9, Eq. [13] and [14] should read

$$V = \int_0^{\ell} \int_0^{2\pi} \left\{ \frac{2}{3} E(1-\lambda) \left[h\alpha + \frac{h^3}{12}\beta \right] \right.$$
$$\left. + \frac{2\lambda E e_s}{\sqrt{3}} \left[\frac{(2\beta z + \gamma)\sqrt{\alpha + 2\gamma + z^2\beta}}{4\beta} + \frac{4\alpha\beta - \delta^2}{8\beta} \frac{1}{\sqrt{\beta}} \sinh^{-1}\left(\frac{2\beta z + \delta}{\sqrt{4\alpha\beta - \gamma^2}} \right) \right]_{-\frac{\ell}{2}}^{\frac{\ell}{2}} \right\} a\, d\varphi\, dx \qquad [13]$$

α and β are correct as shown in report and

$$\gamma(z,\varphi) = -\left(\frac{\partial w}{\partial x}\right)^2\left(\frac{\partial^2 w}{\partial x^2}\right) - \frac{1}{a^2}\frac{\partial^2 w}{\partial \varphi}\left(\frac{\partial w}{a\partial \varphi}\right)^2 + \frac{2}{a^2}\frac{\partial^2 w}{\partial \varphi^2}\frac{w}{a} - \frac{1}{2}\left(\frac{\partial w}{\partial x}\right)^2\frac{1}{a^2}\frac{\partial^2 w}{\partial \varphi^2} - \frac{1}{2}\left(\frac{\partial^2 w}{\partial x^2}\right)\left(\frac{\partial w}{a\partial \varphi}\right)^2 + \frac{w}{a}\frac{\partial^2 w}{\partial x^2} \qquad [14]$$
$$- \frac{1}{a}\frac{\partial^2 w}{\partial x \partial \varphi}\frac{\partial w}{\partial x}\frac{\partial w}{a\partial \varphi}$$

Omit footnote at bottom of p. 9

p. 10, Eq. [13a] should read

$$V_e = \int_0^{\ell} \int_0^{2\pi} \frac{E}{2(1-\nu^2)} \left[h\alpha_e(z,\varphi) + \frac{h^3}{12}\beta_e(z,\varphi) \right] a\, dz\, d\varphi \qquad [13a]$$

p. 11, Eq. [17] and [18] should read

$$V = 2\pi a \int_0^{\ell} \left\{ \frac{2}{3} E(1-\lambda) \left[h\alpha + \frac{h^3}{12}\beta \right] \right.$$
$$\left. + \frac{2\lambda E e_s}{\sqrt{3}} \left[\frac{(2\beta z + \gamma)\sqrt{\alpha + 2\gamma + z^2\beta}}{4\beta} + \frac{4\alpha\beta - \delta^2}{8\beta} \frac{1}{\sqrt{\beta}} \sinh^{-1}\left(\frac{2\beta z + \delta}{\sqrt{4\alpha\beta - \gamma^2}} \right) \right]_{-\frac{\ell}{2}}^{\frac{\ell}{2}} \right\} dz \qquad [17]$$

α and β are correct as shown in report and

$$\gamma(z) = -\left(\frac{\partial w}{\partial x}\right)^2\left(\frac{\partial^2 w}{\partial x^2}\right) + \frac{w}{a}\frac{\partial^2 w}{\partial x^2} \qquad [18]$$

Now define a new variable $x' = \frac{x}{\ell}$ and let $w = w_0 f(x',\varphi)$ then the integrals can be written in terms of dimensionless quantities as follows:

We first combine the elastic and plastic cases in one set of integrals noting that

a. For the elastic case $\lambda = 0$
b. For the plastic case $\nu = \frac{1}{2}$

The combined integrals are then

$$V = \frac{E(1-\lambda)h a \ell}{2(1-\nu^2)} \int_0^1 \int_0^{2\pi} \frac{1}{a}\,dx'd\varphi + \frac{E(1-\lambda)}{2(1-\nu^2)} \frac{h}{3} a\ell \int_0^1 \int_0^{2\pi} \bar{\beta}\,dx'd\varphi$$
$$+ \frac{\lambda E e_s h a \ell}{\sqrt{3}} \int_0^1 \int_0^{2\pi} \left\{ \left[\frac{(2\bar{\beta} + \bar{\gamma})\sqrt{\bar{\alpha} + \bar{\gamma} + \bar{\beta}}}{4\bar{\beta}} + \frac{(4\bar{\alpha}\bar{\beta} - \bar{\delta}^2)}{8\bar{\beta}} \frac{1}{\sqrt{\bar{\beta}}} \sinh^{-1}\left(\frac{2\bar{\beta} + \bar{\delta}}{\sqrt{4\bar{\alpha}\bar{\beta} - \bar{\gamma}^2}} \right) \right] \right.$$
$$\left. - \left[\frac{(-2\bar{\beta} + \bar{\delta})\sqrt{\bar{\alpha} - \bar{\delta} + \bar{\beta}}}{4\bar{\beta}} + \frac{(4\bar{\alpha}\bar{\beta} - \delta^2)}{8\bar{\beta}} \frac{1}{\sqrt{\bar{\beta}}} \sinh^{-1}\left(\frac{-2\bar{\beta} + \bar{\delta}}{\sqrt{4\bar{\alpha}\bar{\beta} - \delta^2}} \right) \right] \right\} dx'd\varphi \qquad [40]$$

where

(over)

$$\bar{\alpha}(x';\varphi) = \left(\frac{w_0}{a}\right)^4\left(\frac{a}{l}\right)^4 \frac{1}{4}\left(\frac{\partial f}{\partial x'}\right)^4 + \left(\frac{w_0}{a}\right)^4\left(\frac{a}{l}\right)^2 \frac{1}{2}\left(\frac{\partial f}{\partial x'}\right)^2\left(\frac{\partial f}{\partial \varphi}\right)^2 - \nu\left(\frac{w_0}{a}\right)^3\left(\frac{a}{l}\right)^2 f\left(\frac{\partial f}{\partial x'}\right)^2$$

$$+ \frac{1}{4}\left(\frac{w_0}{a}\right)^4\left(\frac{\partial f}{\partial \varphi}\right)^4 - \left(\frac{w_0}{a}\right)^3 f\left(\frac{\partial f}{\partial \varphi}\right)^2 + \left(\frac{w_0}{a}\right)^2 f^2$$

$$\bar{\gamma}(x';\varphi) = \frac{h}{2}\gamma = -\left(\frac{w_0}{a}\right)^3\left(\frac{a}{l}\right)^4\frac{h}{2a}\left(\frac{\partial f}{\partial x'}\right)^2\left(\frac{\partial^2 f}{\partial x'^2}\right) - \left(\frac{w_0}{a}\right)^3\frac{h}{2a}\left(\frac{\partial^2 f}{\partial \varphi^2}\right)\left(\frac{\partial f}{\partial \varphi}\right)^2$$

$$+ 2\left(\frac{w_0}{a}\right)^2\frac{h}{2a}\frac{\partial^2 f}{\partial \varphi^2}f - \nu\left(\frac{w_0}{a}\right)^3\left(\frac{a}{l}\right)^2\frac{h}{2a}\left(\frac{\partial f}{\partial x'}\right)^2\left(\frac{\partial^2 f}{\partial \varphi^2}\right)$$

$$- \nu\left(\frac{w_0}{a}\right)^3\left(\frac{a}{l}\right)^2\frac{h}{2a}\left(\frac{\partial^2 f}{\partial x'^2}\right)\left(\frac{\partial f}{\partial \varphi}\right)^2 + 2\nu\left(\frac{w_0}{a}\right)^2\left(\frac{a}{l}\right)^2\frac{h}{2a}f\left(\frac{\partial^2 f}{\partial x'^2}\right)$$

$$- \frac{4(1-\nu)}{2}\left(\frac{w_0}{a}\right)^3\left(\frac{a}{l}\right)^2\frac{h}{2a}\left(\frac{\partial^2 f}{\partial x\partial\varphi}\right)\left(\frac{\partial f}{\partial x'}\right)\left(\frac{\partial f}{\partial \varphi}\right)$$

$$\bar{\beta}(x';\varphi) = \left(\frac{h}{2}\right)^2\beta = \left(\frac{w_0}{a}\right)^2\left(\frac{a}{l}\right)^4\left(\frac{h}{2a}\right)^2\left(\frac{\partial^2 f}{\partial x'^2}\right)^2 + 2\nu\left(\frac{w_0}{a}\right)^2\left(\frac{h}{2a}\right)^2\left(\frac{a}{l}\right)^2\left(\frac{\partial^2 f}{\partial x'^2}\right)\left(\frac{\partial^2 f}{\partial \varphi^2}\right)$$

$$+ \left(\frac{w_0}{a}\right)^2\left(\frac{h}{2a}\right)^2\left(\frac{\partial^2 f}{\partial \varphi^2}\right)^2 + 2(1-\nu)\left(\frac{w_0}{a}\right)^2\left(\frac{a}{l}\right)^2\left(\frac{h}{2a}\right)^2\left(\frac{\partial^2 f}{\partial x'\partial\varphi}\right)^2$$

[40a]

Thus the integrals are dimensionless quantities which are functions of the dimensionless ratios w_0/a, a/l, $h/2a$

Equation [40] is the form that the electronic computer will use to compute the integrals numerically for a given shape $f(x';\varphi)$, given parameters a/l, $h/2a$ and a series of values of w_0/a.

It should be noted that all expressions in the report pertaining to the plastic region are correct if $c_i \gg c_s$. This condition holds for a large number of cases in view of the fact that we are concerned with large deformations and large plastic strains. If this condition is relaxed then expression [10] must be rewritten as

$$V = \int_{\Sigma} \left[\overbrace{\int_0^{e_s} \sigma_i \, de_i}^{elastic} + \overbrace{\int_{e_s}^{e_i} \sigma_i \, de_i}^{plastic} \right] d\Sigma$$

$$= \int_0^{\ell} \int_0^{2\pi} \int_{-\frac{\ell}{2}}^{\frac{\ell}{2}} \left[\frac{E e_i^2}{2}(1-\lambda) + E\lambda e_s e_i \right] a \, d\varphi \, dx \, dz - \frac{E\lambda e_s^2}{2} 2\pi a \ell h$$

The last term must be subtracted from all V expressions if we are dealing with small plastic deformations (i.e. if $c_i \approx c_s$)
In the example worked out in the report, eq. [32] would then become

$$V = 2\pi E e_s h \ell \left(\frac{w_0}{\sqrt{3}} \right) - 2\pi E e_s h \ell \left(\frac{e_s a}{2} \right)$$

so that the new term is negligible if $\frac{w_0}{\sqrt{3}} \gg \frac{e_s a}{2}$.

For very thin shells undergoing very large plastic deformations if $\lambda = 1$,

$$V \approx \int_0^{\ell} \int_0^{2\pi} \frac{2E e_s}{\sqrt{3}} h \sqrt{\alpha(x,\phi)} \, a \, d\varphi \, dx \qquad \text{(see [16])}$$

For some physical cases $\frac{w_0}{a} \ll 1$ $\frac{a}{\ell} \ll 1$ so that

$$\alpha \approx \left(\frac{w_0}{a} \right)^2 f^2$$

Thus for the perfectly plastic case

$$V \approx \frac{2E e_s h w_0 \ell}{\sqrt{3}} \int_0^{\ell} \int_0^{2\pi} f(z,\phi) \, d\varphi \, dx' \qquad ; \quad E e_s = \sigma_s$$

ABSTRACT

A theory for the dynamic elasto-plastic deformations of thin shells
is presented. The deformation is obtained by equating the energy
input to the energy absorbed by the structure. Large deflections
and strain hardening are considered. The theory is applied to the
case of a shell undergoing axially symmetric collapse.

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ACKNOWLEDGEMENTS

LIST OF SYMBOLS

$$\sigma_i = \frac{\sqrt{2}}{2}\sqrt{(\sigma_x - \sigma_y)^2 + (\sigma_y - \sigma_z)^2 + (\sigma_z - \sigma_x)^2 + 6(\tau_{xy}^2 + \tau_{yz}^2 + \tau_{zx}^2)}$$

(Octahedral shear stress is $\frac{\sqrt{2}}{3}\sigma_i$)

$$e_i = \frac{\sqrt{2}}{3}\sqrt{(\epsilon_x - \epsilon_y)^2 + (\epsilon_y - \epsilon_z)^2 + (\epsilon_z - \epsilon_x)^2 + \frac{3}{2}(\gamma_{xy}^2 + \gamma_{yz}^2 + \gamma_{zx}^2)}$$

$\sigma_x, \sigma_y, \sigma_z, \tau_{xy}, \tau_{yz}, \tau_{zx}$ six components of stress; σ's are direct stresses - τ's are shear stresses

$\epsilon_x, \epsilon_y, \epsilon_z, \gamma_{xy}, \gamma_{yz}, \gamma_{zx}$ six components of strain; ϵ's are direct strains - γ's are shear strains

θ $= \epsilon_x + \epsilon_y + \epsilon_z$

K bulk modulus of material

Σ volume of the body

$\epsilon_1, \epsilon_2, \gamma$ strains in the midplane of the shell

z radial distance from the midplane of the shell to any element in the shell

x, ϕ cylindrical coordinates describing the position of the element along the length (x) and around the periphery (ϕ)

y $= a\phi$

a radius of the midplane of the shell (mean radius of the shell)

κ_1, κ_2, τ curvatures (κ) and twist (τ) associated with bending and twisting of the shell element

u, v, w displacements of the shell element; u - longitudinal, v - tangential, w - radial

E modulus of elasticity of shell material

ν Poisson's ratio for shell material (assumed ½ for plastic region)

$\omega(e_i)$ function of e_i

e_s strain at which yielding in pure tension would occur

σ_s yield stress in pure tension

λ parameter involved in the plastic-linear hardening stress strain law; it is the slope of the σ_i-e_i curve in the plastic region

h thickness of the shell

ℓ	length of the shell
V	energy absorbed by shell
dm	element of mass
$\dot{w}$	velocity of mass in radial direction
I	impulse per unit mass
ρ	mass density of shell material
T	initial Kinetic energy put into shell by pressure pulse
I_t	total impulse
I_o	amplitude I
p	pressure applied to shell
w_o	amplitude of w
p_o	amplitude of p

I. Introduction

Shell theory, especially the dynamic plasticity of shells, is in such a state at the present time that one can only hope to obtain approximate solutions to the problems that are of practical interest today. Very little work has been done on the dynamic plasticity of shells due in part to the mathematical complexities of the theory and probably due to the lack of experimental evidence with which to check the results of the theoretical developments. Fortunately the electronic computer will enable us to overcome some of the mathematical difficulties.

Extensive use is being made of shells in missiles and submarines. Therefore more complete experiments are being conducted on these structures and more experimental evidence is becoming available to those working in the field of dynamic elasticity and plasticity. More theoretical development on this problem is needed and it is for this purpose that the present report has been written. This report will contain an approximate method for predicting large elastic and plastic deformations of shells under static and dynamic loads.

II. Theory

A. Energy relations

Assume that the shell is exposed to an impulsive load of short duration which imparts an initial velocity to the structure. The problem is then equivalent to one in which the structure has an initial Kinetic energy.[1,2] If the energy absorbed by the shell is plotted against the lateral deflection in both the elastic and plastic regions, then according to a previous reference[3] there can be a region of instability in the elastic region which corresponds to snap buckling of the structure. If no instabilities occur in

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[1] Witmer, et al "Response of Plates and Shells to Intense External Loads of Short Duration," WADD Technical Report 60-433, Apr. 1960 (Mass. In. Tech.)

[2] Pian, T. H.H., "Dynamic Response of Thin Shell Structures," Plasticity, Proc. Sec. Symp. on Naval Struct. Mech., 1960, p. 443.

[3] Radkowski, P.P. et al, "Studies on the Dynamic Response of Shell Structures and Materials to a Pressure Pulse," Avco Corp., 1961, AFSWC-TR-61-31 (II)

the elastic region, when the structure goes plastic there could be a plastic instability point. If this point does not exist and the structur is loaded further then there should be a point at which a slight increase in energy will correspond to a relatively large increase in deflection. This should happen when the structure is about to collapse.

The initial kinetic energy can be determined in terms of the applied impulse. This impulse will in general turn out to be a nonlinear function of the lateral displacement if large strains occur and if the corresponding large deflection expressions are employed. It will be assumed that all the initial kinetic energy that is imparted to the shell by the impulse goes into energy of deformation. The final deformation will occur when the shell comes to rest so that the final kinetic energy is zero.

Schuman[4] has conducted experiments on a large number of aluminum and steel cylinders subject to nonuniform blast loading. He found two characteristic types of failure which are shown in Figure 1.
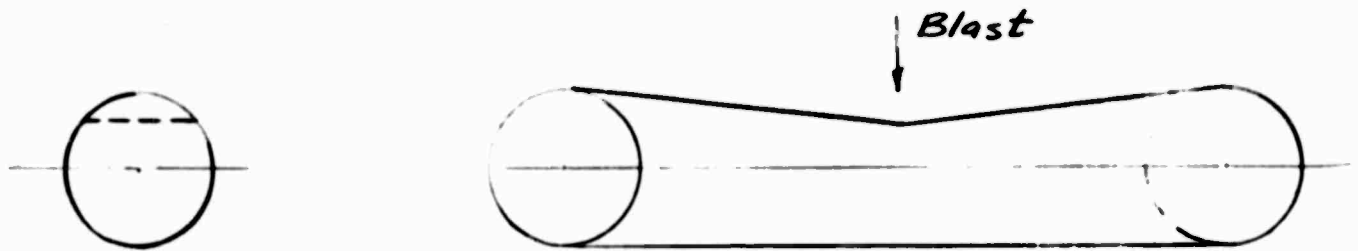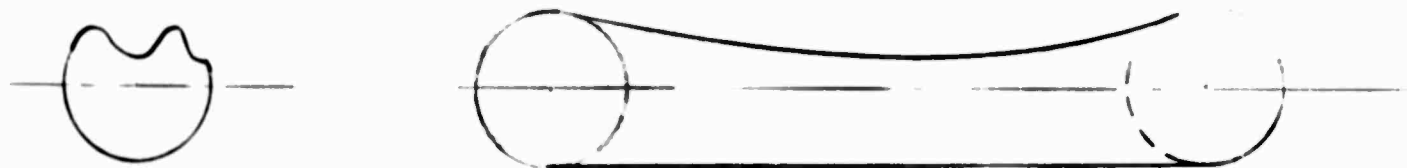

Fig. 1a. Failure Type 1

Fig. 1b. Failure Type 2

[4] Unpublished data on tests performed by William Schuman at the Ballistic Research Laboratories at Aberdeen Proving Ground.

It seems that failure type 1 is a collapse with a failure hinge at the center of the cylinder. Failure type 2 resembles a buckling type failure. Whether or not these types of failure fit into the catagories of plastic collapse and buckling instability remains to be seen. It is proposed to use these general failure shapes found experimentally as the deformation patterns in the energy expression. In order to determine the final plastic deformation due to a given impulse the energy-deflection relation will first be computed. The initial kinetic energy will then be equated to the energy of deformation absorbed by the shell. The applied impulse can be written in terms of the initial kinetic energy as will be shown later in the report. The final result will be a curve of applied impulse as a function of final deflection. This curve will implicitly contain all the nonlinearities in the strain-deflection relations and all the details of the elastic-plastic stress-strain law. The regions of buckling and collapse can be determined from the energy-deflection curve as shown in Fig. 2
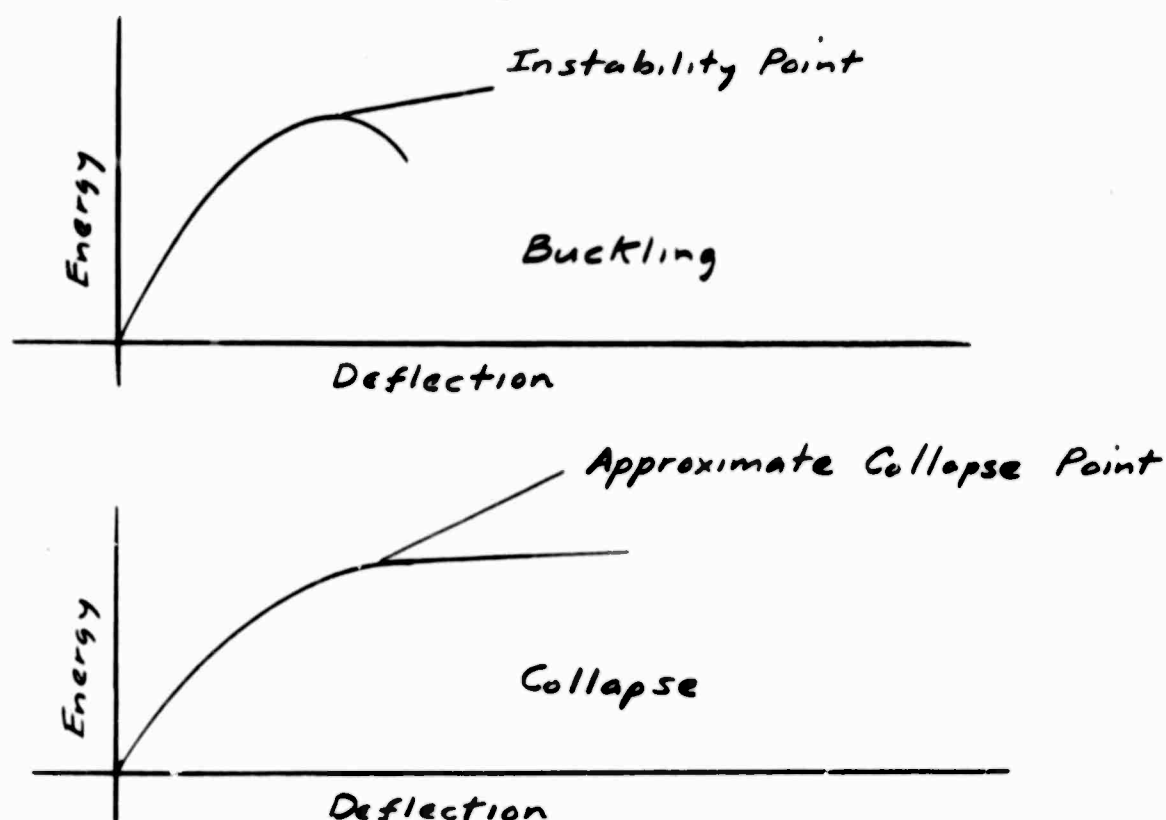

Fig. 2 Buckling and Collapse Curves

The deformation energy (or work done by the internal forces) per unit volume of an elastic-plastic body can be written[5]

[5]Iliouchine, "Plasticite (Translated from the original Russian into French), Edition Eyrolles, 1956, p. 98.

$$V = \int_0^{e_i} \sigma_i \, de_i + \frac{K\theta^2}{2}$$

$$\sigma_i = \frac{\sqrt{2}}{2}\sqrt{(\sigma_x - \sigma_y)^2 + (\sigma_y - \sigma_z)^2 + (\sigma_z - \sigma_x)^2 + 6(\tau_{xy}^2 + \tau_{yz}^2 + \tau_{zx}^2)}$$

$$e_i = \frac{\sqrt{2}}{3}\sqrt{(\epsilon_x - \epsilon_y)^2 + (\epsilon_y - \epsilon_z)^2 + (\epsilon_z - \epsilon_x)^2 + \frac{3}{2}(\gamma_{xy}^2 + \gamma_{yz}^2 + \gamma_{zx}^2)} \qquad [1]$$

$$\theta = \epsilon_x + \epsilon_y + \epsilon_z \qquad\qquad K = bulk\ modulus$$

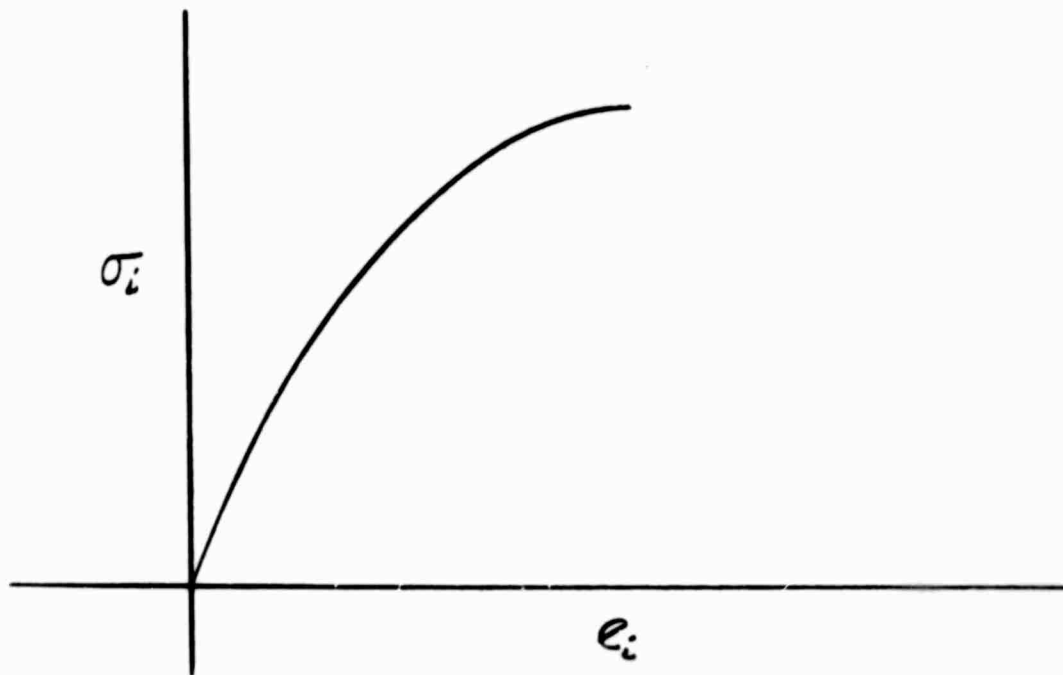The curve of σ_i vs. e_i describes the stress-strain law of the material as shown in Fig. 3

Fig. 3 General Stress-Strain Law

Assuming an incompressible material $(\theta = 0)$ and particularizing our analysis to a thin shell

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$$V = \int_{\Sigma} \left[\int_0^{e_i} \sigma_i \, de_i \right] dZ$$

$$\sigma_i = \sqrt{\sigma_x^2 - \sigma_x \sigma_y + \sigma_y^2 + 3\tau_{xy}^2}$$

[2]

$$e_i = \frac{2}{\sqrt{3}} \sqrt{\epsilon_x^2 + \epsilon_x \epsilon_y + \epsilon_y^2 + \frac{1}{4}\delta_{xy}^2}$$
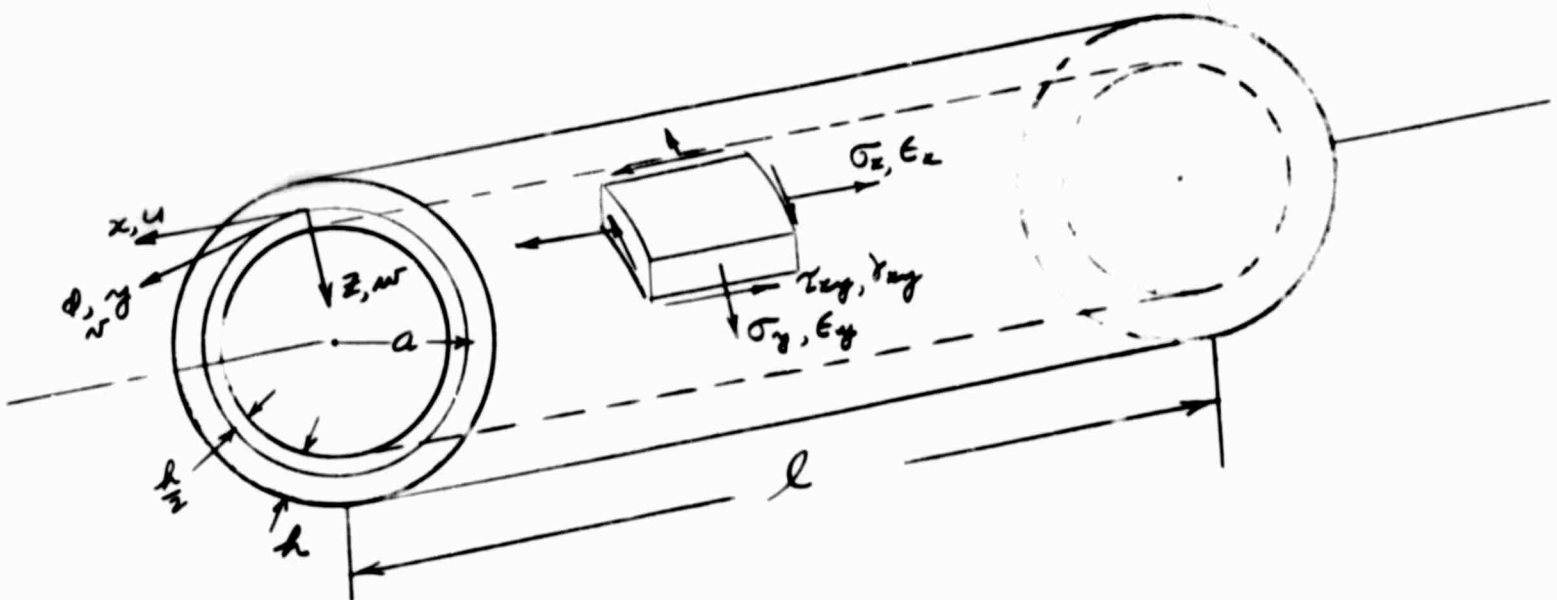
The notation is as shown in Fig. 4

Fig. 4 The Shell

The expressions for the strains are[6]

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[6]Love, "Mathematical Theory of Elasticity," Fourth Edition, Dover Publications, 1944, p. 529.

$$\epsilon_x = \epsilon_1 - z\kappa_1 \quad , \quad \epsilon_y = \epsilon_2 - z\kappa_2 \quad , \quad \gamma_{xy} = \gamma - 2z\tau \qquad [3]$$

where ϵ_1, ϵ_2, γ are the midsurface strains, z is the radial distance from the midsurface to any element within the thickness of the shell and κ_1, κ_2, τ are the curvatures and twist. The values of ϵ_1, ϵ_2, γ are (allowing large deflections)[7]

$$\epsilon_1 = \frac{\partial u}{\partial x} + \frac{1}{2}\left(\frac{\partial w}{\partial x}\right)^2 \quad , \quad \epsilon_2 = \frac{1}{a}\frac{\partial v}{\partial\phi} + \frac{1}{2}\left(\frac{\partial w}{a\partial\phi}\right)^2 - \frac{w}{a}$$

$$\gamma = \frac{\partial v}{\partial x} + \frac{1}{a}\frac{\partial u}{\partial\phi} + \frac{\partial w}{\partial x}\frac{\partial w}{a\partial\phi} \qquad [4]$$

and the curvatures and twist are[8]

$$\kappa_1 = \frac{\partial^2 w}{\partial x^2} \quad , \quad \kappa_2 = \frac{1}{a^2}\frac{\partial^2 w}{\partial\phi^2} + \frac{1}{a^2}\frac{\partial v}{\partial\phi}$$

$$\tau = \frac{1}{a}\frac{\partial^2 w}{\partial x\partial\phi} + \frac{1}{a}\frac{\partial v}{\partial x} \qquad [5]$$

A general deformation type elastic-plastic stress-strain law can be written[5] (assuming a Poissons ratio of ½)

$$\sigma_x = \frac{4}{3}\frac{\sigma_i}{e_i}\left(\epsilon_x + \frac{1}{2}\epsilon_y\right)$$

$$\sigma_y = \frac{4}{3}\frac{\sigma_i}{e_i}\left(\epsilon_y + \frac{1}{2}\epsilon_x\right) \qquad [6]$$

$$\tau_{xy} = \frac{1}{3}\frac{\sigma_i}{e_i}\gamma_{xy}$$

where

$$\frac{\sigma_i}{e_i} = E\left[1 - \omega(e_i)\right] \qquad [7]$$

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[7] Fung and Sechler, "Instability of Thin Elastic Shells," Proc. of the First Symp. on Naval Struct. Mech., 1958, p. 118.

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in which $\omega(e_i) = 0$ for the elastic region. Now neglect all terms in the strains which contain u and ν, assuming that they are small compared to w and its derivatives[*]

Then

$$\epsilon_x = \frac{1}{2}\left(\frac{\partial w}{\partial x}\right)^2 - z\frac{\partial^2 w}{\partial x^2} \quad , \quad \epsilon_y = \frac{1}{2}\left(\frac{\partial w}{a\partial\varphi}\right)^2 - \frac{w}{a} - \frac{z}{a^2}\frac{\partial^2 w}{\partial\varphi^2}$$

$$\gamma_{xy} = \frac{\partial w}{\partial x}\frac{\partial w}{a\partial\varphi} - 2\frac{z}{a}\frac{\partial^2 w}{\partial x \partial\varphi} \tag{8}$$

Consider an elastic-linear hardening incompressible material[**] which has a stress strain law as shown in Fig. 5

$$\overline{\sigma_s} = E e_s$$

$\overline{\sigma_s}$ = yield stress in pure tension
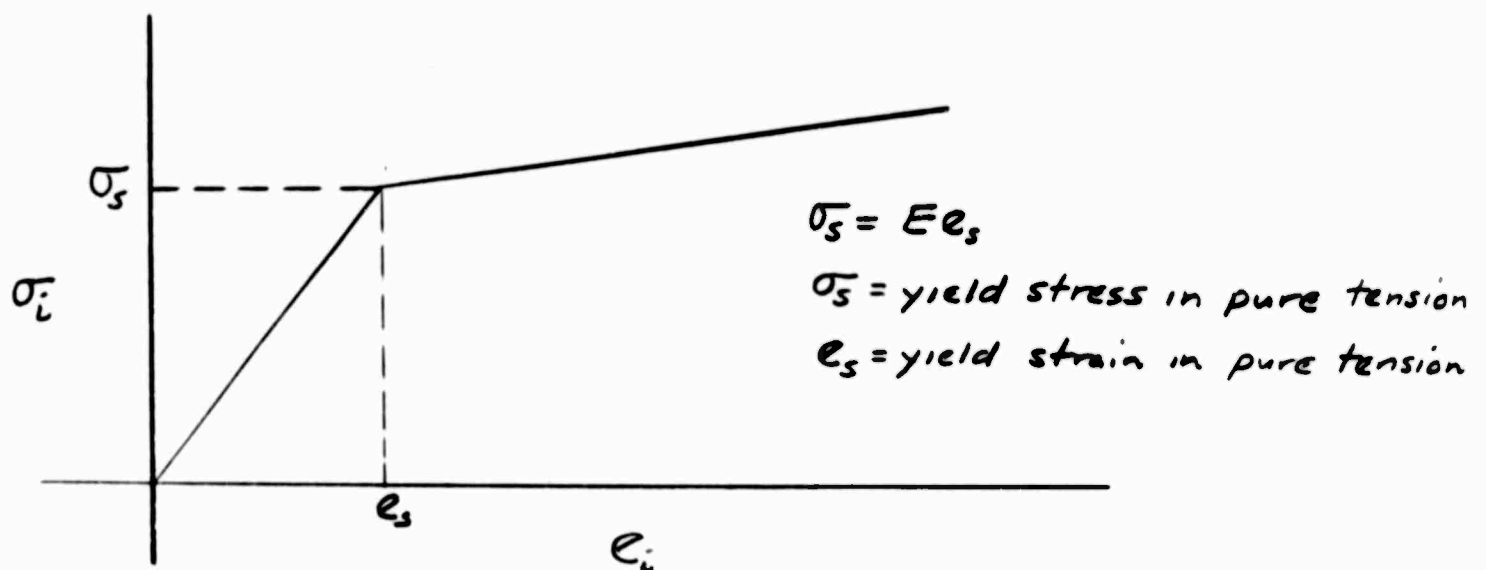
e_s = yield strain in pure tension

Fig. 5 Elastic-Linear Hardening Law

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[*]This assumption seems to have been excellent in predicting very large strains in panels, see J. Greenspon, J. Aero/Space Sciences, 27, 5, May, 1960.

[**]Aluminum and steel both come close to being linear hardening materials - see ANC-5 "Strength of Metal Aircraft Elements," U. S. Government Printing Office.

$$\text{Elastic Case} \qquad \omega(c_i) = 0 \qquad \text{for} \quad c_i < c_s$$

$$\text{Plastic Case} \qquad \omega(c_i) = \lambda\left(1 - \frac{c_s}{c_i}\right) \text{ for } c_i > c_s \qquad [9]$$

$$\lambda = 1 - \frac{1}{E}\frac{d\sigma_i}{dc_i}$$

It is planned to use this stress-strain law. The energy-deflection relation will then be computed and a search will be made for regions of instability in the elastic and plastic regions and a region of collapse in the plastic regime. If the value of energy corresponding to instability is smaller than the amount corresponding to collapse, the structure will buckle first; however if the collapse energy is smaller than the energy corresponding to instability, the structure will collapse first.

Going back to the general expression for the energy, and substituting the stress strain law of eq. [9]

$$V = \int_{\Sigma}\left[\int_0^{c_i} \sigma_i\, dc_i\right] d\Sigma$$

$$= \int_0^\ell \int_0^{2\pi} \int_{-\frac{k}{2}}^{\frac{k}{2}} \int_0^{c_i} E c_i \left[1 - \lambda\left(1 - \frac{c_s}{c_i}\right)\right] dc_i\, a\, d\varphi\, dx\, dz \qquad [10]$$

$$V = \int_0^\ell \int_0^{2\pi} \int_{-\frac{k}{2}}^{\frac{k}{2}} \left[\frac{E c_i^2}{2}(1-\lambda) + E\lambda c_s c_i\right] a\, d\varphi\, dx\, dz \qquad [11]$$

Substituting the expression for c_i from [2]

$$V = \int_0^\ell \int_0^{2\pi} \int_{-\frac{\ell}{2}}^{\frac{\ell}{2}} \left[\frac{E}{2}(1-\lambda)\frac{4}{3}\left(\epsilon_x^2 + \epsilon_x\epsilon_y + \epsilon_y^2 + \frac{1}{4}\gamma_{xy}^2\right) \right. \qquad [12]$$
$$\left. + \frac{E\lambda c_s}{\sqrt{3}} 2\sqrt{\epsilon_x^2 + \epsilon_x\epsilon_y + \epsilon_y^2 + \frac{1}{4}\gamma_{xy}^2}\right] a\, d\varphi\, dx\, dz$$

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Using the strain expressions [8] and integrating with respect to z

$$V = \int_0^\ell \int_0^{2\pi} \left\{ \frac{2}{3} E(1-\lambda)\left[h\,\alpha(z,\phi) + \frac{h^3}{4}\beta(z,\phi) \right] \right.$$
$$\left. + \frac{2}{3}\frac{2\lambda E c_s}{\sqrt{3}}\left[\frac{(\alpha(z,\phi)+z\beta(z,\phi))^{3/2}}{\beta(z,\phi)} \right]_{z=-\frac{h}{2}}^{z=\frac{h}{2}} \right\} a\,d\phi\,dz \qquad [13]$$

$$\alpha(x,\phi) = \frac{1}{4}\left(\frac{\partial w}{\partial x}\right)^4 + \frac{1}{2}\left(\frac{\partial w}{\partial x}\right)^2\left(\frac{\partial w}{a\partial\phi}\right)^2 - \frac{1}{2}\frac{w}{a}\left(\frac{\partial w}{\partial x}\right)^2 + \frac{1}{4}\left(\frac{\partial w}{a\partial\phi}\right)^4 - \frac{w}{a}\left(\frac{\partial w}{a\partial\phi}\right)^2 + \frac{w^2}{a^2}$$

$$\beta(x,\phi) = \left(\frac{\partial^2 w}{\partial x^2}\right)^2 + \frac{1}{a^2}\frac{\partial^2 w}{\partial x^2}\frac{\partial^2 w}{\partial\phi^2} + \frac{1}{a^4}\left(\frac{\partial^2 w}{\partial\phi^2}\right)^2 + \frac{1}{a^2}\left(\frac{\partial^2 w}{\partial x\partial\phi}\right)^2 \qquad [14]$$

The strain terms in equation [8] which involve z are associated with bending, the other terms are the membrane strains. For very thin shells sometimes the bending strains can be neglected and the resulting energy expression will be*

$$V = \int_0^\ell \int_0^{2\pi} \left\{ \frac{2}{3} E(1-\lambda)\left[h\,\alpha(x,\phi) \right] \right.$$
$$\left. + \frac{2\lambda E c_s}{\sqrt{3}} h\sqrt{\alpha(x,\phi)} \right\} a\,d\phi\,dx \qquad [15]$$

Some materials can be considered perfectly plastic. For this case $\lambda = 1$ in the plastic region and

$$V_p = \int_0^\ell \int_0^{2\pi} \frac{2\lambda E c_s}{\sqrt{3}} h\sqrt{\alpha(x,\phi)}\, a\,d\phi\,dx \qquad [16]$$

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*Obtained by setting $\beta(x,\phi) = 0$ in [13]; note that L'Hopital's rule must be applied to the second bracket.

The energy of deformation in the elastic region can be obtained from
[13] by letting $\lambda = 0$. However a more accurate expression for the
elastic region can be obtained by starting with the elastic stress
strain law for any Poissons ratio, i.e.

$$\sigma_x = \frac{E}{1-\nu^2} (\epsilon_x + \nu \epsilon_y)$$

$$\sigma_y = \frac{E}{1-\nu^2} (\epsilon_y + \nu \epsilon_x)$$

$$\tau_{xy} = \frac{E}{2(1+\nu)} \gamma_{xy}$$

[6a]

The elastic energy is

$$V_e = \frac{1}{2} \int_\Sigma \frac{E}{1-\nu^2} (\epsilon_x^2 + 2\nu \epsilon_x \epsilon_y + \epsilon_y^2 + \frac{1-\nu}{2} \gamma_{xy}^2) \, d\Sigma$$

[12a]

Thus

$$V_\epsilon = \int_0^\ell \int_0^{2\pi} \frac{E}{2(1-\nu^2)} \left[\hbar \, \alpha_e(x,\phi) + \frac{\hbar^3}{4} \beta_e(x,\phi) \right] a \, dx \, d\phi$$

[13a]

where

$$\alpha_e(x,\phi) = \frac{1}{4} \left(\frac{\partial w}{\partial x} \right)^4 + \frac{\nu}{2} \left(\frac{\partial w}{\partial x} \right)^2 \left(\frac{\partial w}{a \partial \phi} \right)^2 - \nu \left(\frac{\partial w}{\partial x} \right)^2 \frac{w}{a} + \frac{1}{4} \left(\frac{\partial w}{a \partial \phi} \right)^4 - \frac{w}{a} \left(\frac{\partial w}{a \partial \phi} \right)^2$$
$$+ \frac{w^2}{a^2} + \frac{1-\nu}{2} \left(\frac{\partial w}{\partial x} \right)^2 \left(\frac{\partial w}{a \partial \phi} \right)^2$$

[14a]

$$\beta_e(x,\phi) = \left(\frac{\partial^2 w}{\partial x^2} \right)^2 + \frac{2\nu}{a^2} \frac{\partial^2 w}{\partial x^2} \frac{\partial^2 w}{\partial \phi^2} + \frac{1}{a^4} \left(\frac{\partial^2 w}{\partial \phi^2} \right)^2 + \frac{2(1-\nu)}{a^2} \left(\frac{\partial^2 w}{\partial x \partial \phi} \right)^2$$

In some cases the deformation of the shell may be axisymmetric and the deflection w will then be independent of φ . For this case expression [13] becomes

$$V = 2\pi a \int_0^{\ell} \left\{ \frac{2}{3} E(1-\lambda)\left[\ell\alpha(x) + \frac{h^2}{4}\beta(x)\right] \right. $$
$$\left. + \frac{4}{3\sqrt{3}} \lambda E \ell_s \left[\frac{(\alpha(x) + z\beta(x))^{3/2}}{\beta(x)}\right]_{-\frac{h}{2}}^{\frac{h}{2}} \right\} dx \qquad [17]$$

where

$$\alpha(x) = \frac{1}{4}\left(\frac{\partial w}{\partial x}\right)^4 - \frac{1}{2}\frac{w}{a}\left(\frac{\partial w}{\partial x}\right)^2 + \frac{w^2}{a^2}$$

$$\beta(x) = \left(\frac{\partial^2 w}{\partial x^2}\right)^2 \qquad [18]$$

This concludes the basic energy equations. The relation between the kinetic energy input and the applied impulse will now be considered.

B. Impulse-Energy Relation

Let I be the impulse per unit mass applied to the shell. The impulse momentum relation for an elemental mass dm can be written

$$\dot{w}\, dm = I\, dm \qquad [19]$$

where $\dot{w}$ is the lateral velocity imparted to the mass by the impulse*
Thus

$$\dot{w} = I \qquad [20]$$

The kinetic energy imparted to the shell is

---

*Only the lateral velocity $\dot{w}$ is being considered, $\dot{u}$ and $\dot{v}$ are being neglected.

$$T = \int_A \tfrac{1}{2}\, \rho \ell \,\dot{w}^2 dA = \tfrac{1}{2} \int_A \rho \ell \, I^2 dA \qquad [21]$$

where $\rho \ell$ is the mass per unit area and dA is an elemental area. The impulse can vary over the surface of the shell in the same way that the pressure can vary over the surface. Therefore write the impulse as

$$I(x,y) = I_0\, f(x,y) \qquad [22]$$

Thus from [21]

$$T = \tfrac{1}{2} \rho \ell \, I_0^2 \int_A f^2(x,y)\, dA \qquad [23]$$

Equating the initial kinetic energy to the energy of deformation absorbed by the shell the expression for the impulse per unit mass becomes

$$I_0 = \sqrt{ V \frac{2}{\rho \ell \int_A f^2(x,y)\, dA} } \qquad [24]$$

The total impulse on the shell will then be

$$I_t = \int_A I_0\, \rho \ell \, f(x,y)\, dA = \int_A \sqrt{ V \frac{2 \rho \ell}{\int_A f^2(x,y)\, dA} }\; f(x,y)\, dA \qquad [25]$$

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The impulse per unit area will be

$$I_o \rho h = Impulse/unit\,area = \sqrt{V \frac{2\rho h}{\int_A f^2(x,y)\,dA}}$$ [26]

If the impulse is uniform

$$f(x,y) = 1$$

and

$$I_o \rho h = \sqrt{V \frac{\rho h}{\pi a \ell}}$$ [27]

It should be emphasized that the expressions for the impulse hold when the shell has reached its final state of deformation and has come to rest or when an unstable point is reached where the kinetic energy is zero.

C. Static loading

In the case of static loading the energy of deformation is equated to the work done by the applied load. The result will be a curve of load vs deflection. The static buckling or collapse load can then be obtained in a similar fashion as described before; i.e. first the energy deflection relation is plotted and the deflection at which instability or collapse occurs is determined as shown in Fig. 2. The load corresponding to this deflection then defines the static buckling or collapse load. Since the energy-deflection relation will in general be nonlinear and probably will not contain sharp discontinuities, the collapse load will not be well defined. However it is expected that a region of collapse can be determined. The work done by the static load will be

$$W = \int_A p(x,y)\,w\,dA$$ [28]

where p is the applied pressure and w the lateral deflection.

D. A simple application

Although extensive calculations will be done in the near future using the relations derived here, a very simple application will now be considered in which the integrals can be evaluated analytically. Consider the axially symmetric lateral loading of a perfectly plastic shell subjected to both static and dynamic loading which is assumed uniform over the shell. The problem is to estimate the static collapse load and to determine the impulse-deflection relationship in the plastic region.

It has been found experimentally[4] that cylindrical shells under axially symmetric external load usually collapse with a deformation pattern resembling two frustrums of a cone with a hinge circle at the center of the cylinder as shown in Fig. 6
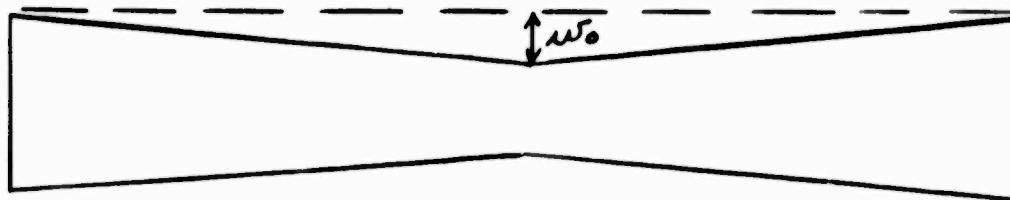

Fig. 6 Collapse Pattern of Cylindrical Shell
under Axially Symmetric Load

According to the sign convention used in this analysis w is positive inward. The axes and deformation pattern are shown in Figure 7.
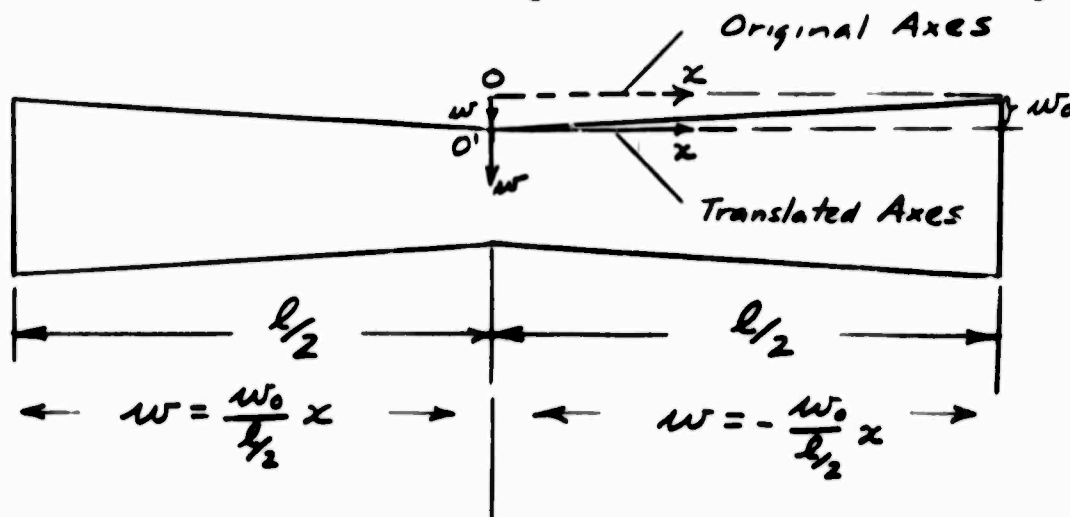

Fig. 7 Deformation Pattern of Shell

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Thus

$$V = \frac{2}{\sqrt{3}} E \ell_s h \cdot 2\pi a \left\{ \int_{-\frac{\ell}{2}}^{0} \sqrt{\frac{1}{4}\left(\frac{w_0}{\ell/2}\right)^4 - \frac{1}{2}\left(\frac{w_0}{\ell/2}\right)^3 \frac{z}{a} + \left(\frac{w_0}{\ell/2}\right)^2 \frac{z^2}{a^2}} \, dz \right.$$

$$\left. + \int_{0}^{\ell/2} \sqrt{\frac{1}{4}\left(\frac{w_0}{\ell/2}\right)^4 + \frac{1}{2}\left(\frac{w_0}{\ell/2}\right)^3 \frac{z}{a} + \left(\frac{w_0}{\ell/2}\right)^2 \frac{z^2}{a^2}} \, dz \right. \qquad [29]$$

Integrating

$$V = \frac{4\pi a E \ell_s h}{\sqrt{3}} \left\{ \frac{1}{8}\left(\frac{w_0}{\ell/2}\right)^3 a + \frac{3}{16}\left(\frac{w_0}{\ell/2}\right)^3 a \sinh^{-1}\frac{1}{\sqrt{3}} \right.$$

$$+ \frac{\ell + \frac{1}{2}\left(\frac{w_0}{\ell/2}\right)a}{2}\sqrt{\frac{1}{4}\left(\frac{w_0}{\ell/2}\right)^4 + \frac{1}{2}\left(\frac{w_0}{\ell/2}\right)^3 \frac{\ell}{2a} + \left(\frac{w_0}{\ell/2}\right)^2 \frac{\ell^2}{4a^2}} \qquad [30]$$

$$\left. + \frac{3}{16}\left(\frac{w_0}{\ell/2}\right)^3 a \sinh^{-1}\left(\frac{\frac{\ell}{a} + \left(\frac{w_0}{\ell/2}\right)}{\sqrt{3}\left(w_0/\ell/2\right)}\right) \right\}$$

Assume that the shell and deformation pattern are such that*

$$\frac{w_0}{\ell/2} << 1 \qquad , \qquad \ell/a >> 1 \qquad [31]$$

Thus the energy expression reduces to

$$V = \frac{2\pi E \ell_s h}{\sqrt{3}} w_0 \ell \qquad [32]$$

For static loading the work done by the external uniform lateral load is

---

*This will be true of the majority of practical cases.

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$$W = 2\pi a \; p_0 \; \frac{w_0 \ell}{2} \tag{33}$$

Equating the internal energy absorbed by the shell to the work done by the external load it is found that

$$P_0 = \frac{2}{\sqrt{3}} \; \frac{E e_s \ell}{a} = \frac{2}{\sqrt{3}} \; \frac{\sigma_s \ell}{a} \tag{34}$$

Expression [34] implies that the load will not be dependent on the deflection throughout the plastic region. Therefore this load will correspond to any deflection in the plastic region and can therefore be termed the static collapse load.

It is interesting to note that the load corresponding to yield in an elastic shell with ordinary hoop tension is

$$(P_0)_e = \frac{\sigma_s \ell}{a} \tag{35}$$

Thus [34] predicts that the static collapse load is about 15% higher than the load at which yielding will start.

For shells in which ℓ/a is not much greater than unity higher order terms in the energy must be included. If second order terms in $w_0/\ell_{1/2}$ are retained in [30] the energy becomes

$$V = \frac{4\pi a E e_s \ell}{\sqrt{3}} \left(\frac{w_0 \ell}{2a} + \frac{1}{4} \frac{w_0^2}{\ell_{1/2}} \right) \tag{36}$$

Therefore the static load-deflection relation in the plastic region will be

$$P_0 = \frac{2E e_s \ell}{\sqrt{3} \; a} \left(1 + \frac{1}{4} \frac{w_0}{\ell_{1/2}} \frac{a}{\ell_{1/2}} \right) \tag{37}$$

This will give a load deflection curve which looks schematically as shown in Fig. 8.
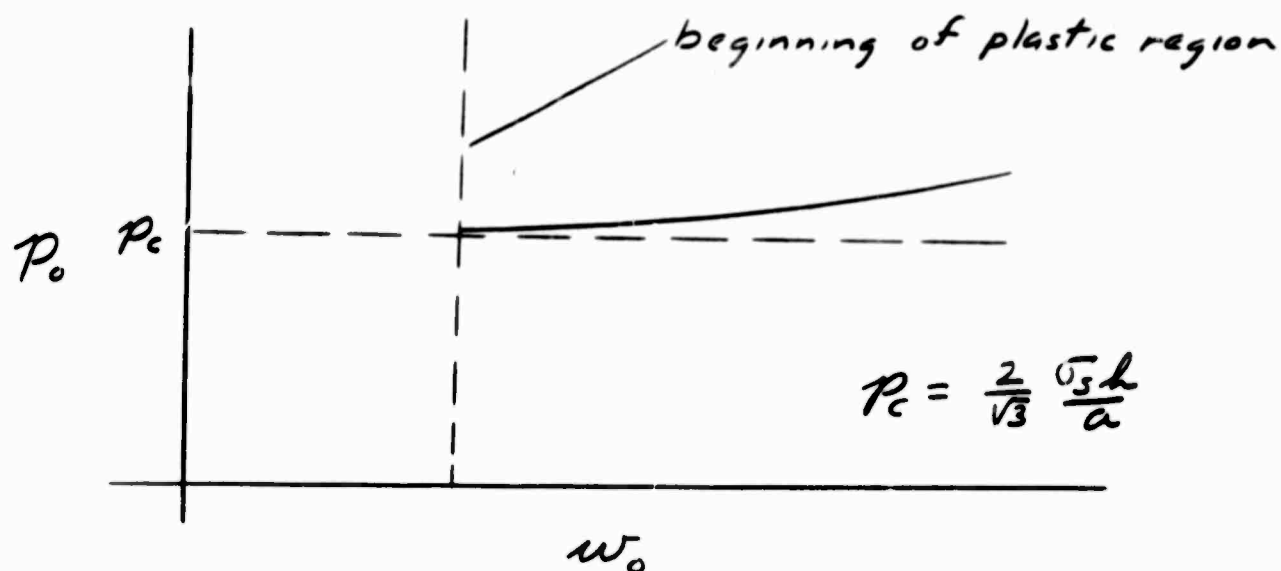
Fig. 8 Load Deflection Curve in Plastic Region
Including Nonlinear Terms

The effect of the nonlinear terms is to stiffen the shell so that
the plastic deflection for a given load will be a definite value.
For the linear case the collapse load would produce any plastic
deflection.

Using relations [32] or [36] with [27] the impulse-deflection curve
in the plastic region can be obtained. A schematic of the results
using these two energy relationships is given in Figure 9.
    with [32]

$$(I_0 \rho h)^2 = \frac{2\pi E e_s h}{\sqrt{3}} w_0 \ell \frac{\rho h}{\pi a \ell} \qquad [38]$$

.. with [36]

$$(I_0 \rho h)^2 = \frac{4\pi a E e_s h}{\sqrt{3}} \left(\frac{w_0 \ell}{2a} + \frac{1}{4} \frac{w_0^2}{\ell_{\frac{1}{2}}} \right) \frac{\rho h}{\pi a \ell} \qquad [39]$$
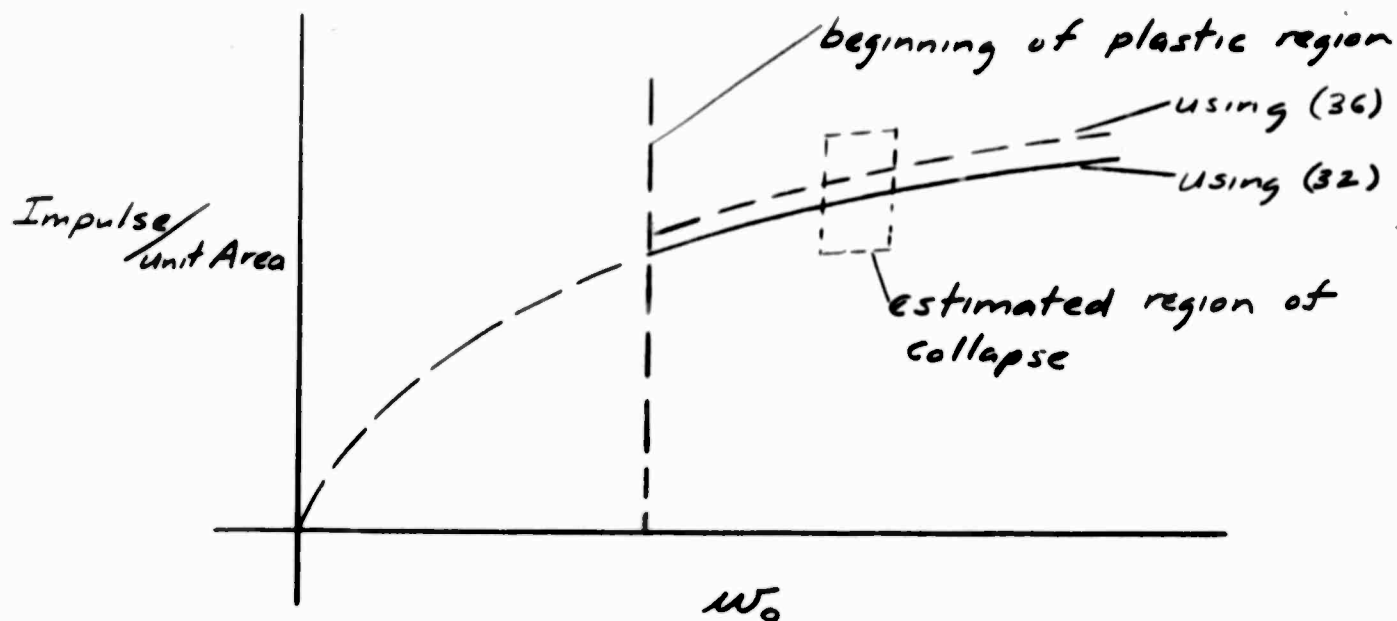
Fig. 9 Impulse Deflection Relation

The static collapse load does not define collapse in a dynamic prob-
lem. In fact if the load is applied dynamically the actual magni-
tude of the load can be much greater than the static collapse load
and still not result in plastic deformation. It is the load-time
relation that is important and the impulse is a lumped parameter
which essentially is a measure of the load time history.

III. Discussion

The great advantage of the energy absorption method as described
here and elsewhere in the literature[1,2] is that many complications
can be considered in the analysis and still result in a tractable
problem. For example it is seen that we have considered here non-
linearity in the strains, strain hardening in the stress-strain
law and nonuniform dynamic loading. This type of theory has one
main disadvantage - we have to assume a deflection pattern. How-
ever it must be realized that even in the more sophisticated plas-
ticity theories assumptions on the pattern must also be made.

Based on careful study of the existing plasticity flow theories of
today it is difficult for this author to see how the important com-
plications such as nonlinearity in the strains and stress-strain law
in addition to nonuniform dynamic loading can be included and result
in a solvable problem. In such plastic dynamic biaxial stress prob-
lems this author sees very little hope for any but the deformation
theories and the energy type procedure for actually getting reasonable
answers in a reasonable time.